# **BRIDGES** CONFERENCE

MATHEMATICS · ART · MUSIC · ARCHITECTURE · CULTURE

AALTO UNIVERSITY HELSINKI, FINLAND

2-3 AUGUST, 2021

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In light of the restrictions and challenges imposed upon us by the ongoing COVID-19 pandemic, Bridges 2021 will be held online.

JO NIEMEYER
RABE VON RANDOW

# DOUBLING THE CUBE - REVISITED

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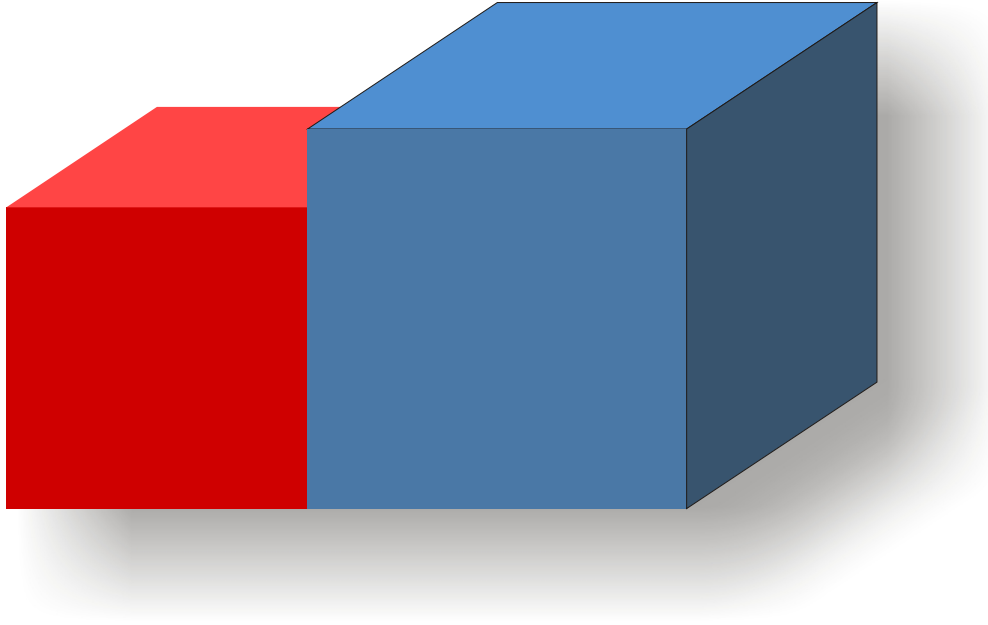
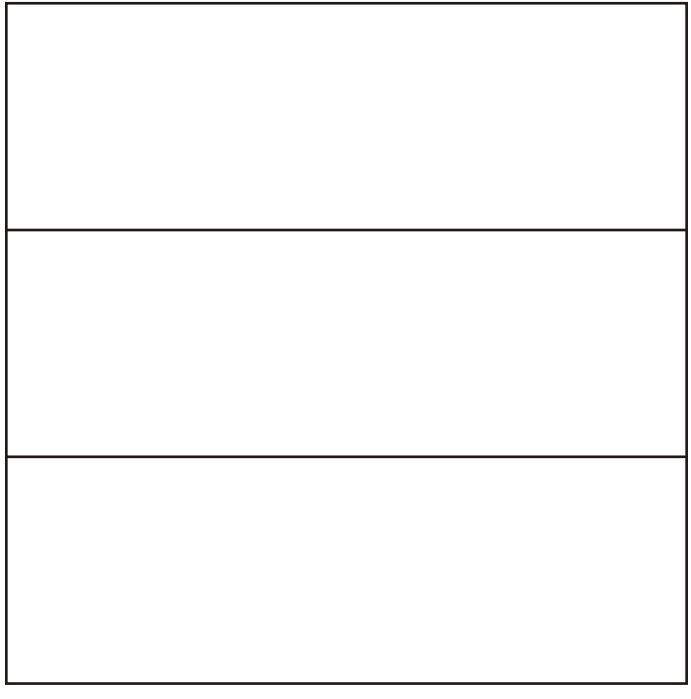
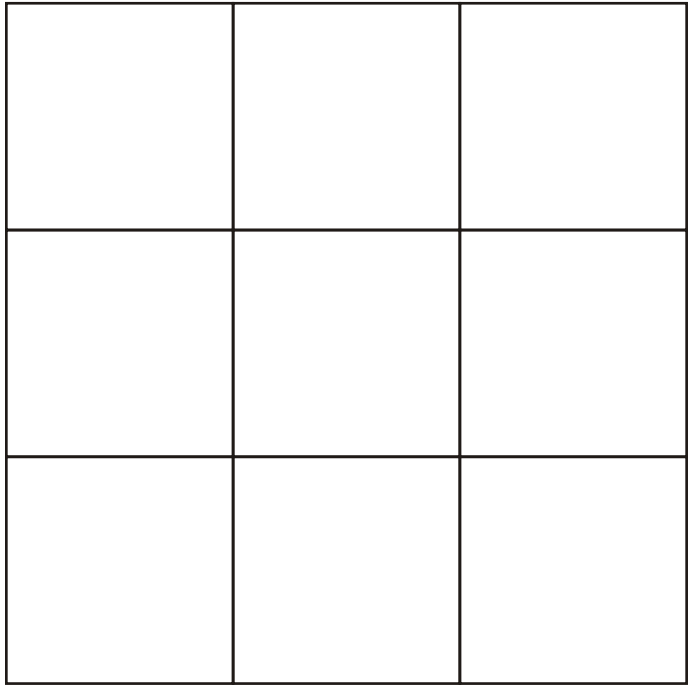
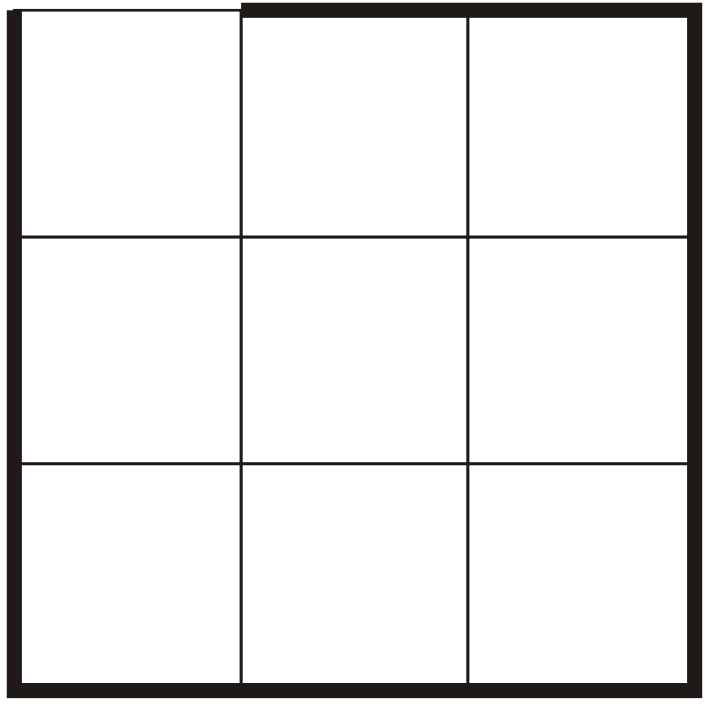
# DOUBLING THE CUBE - REVISITED

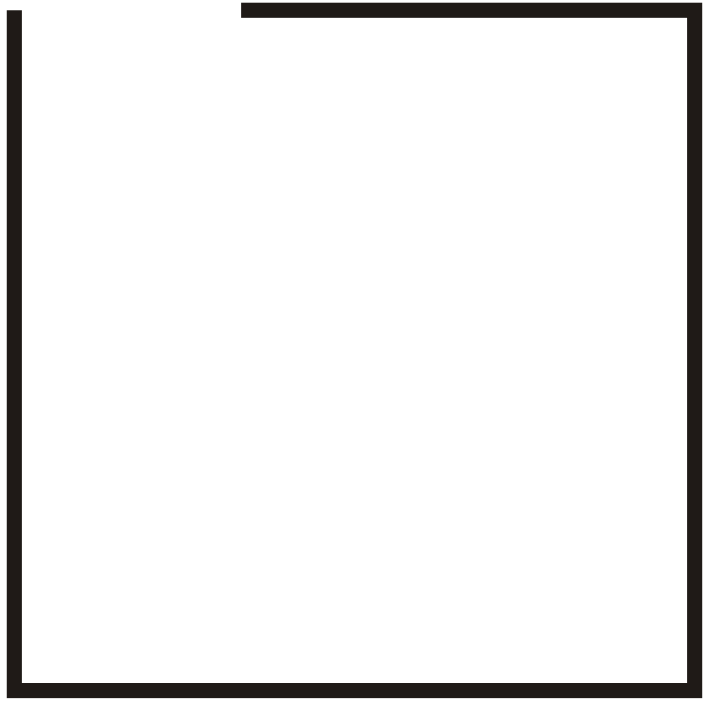


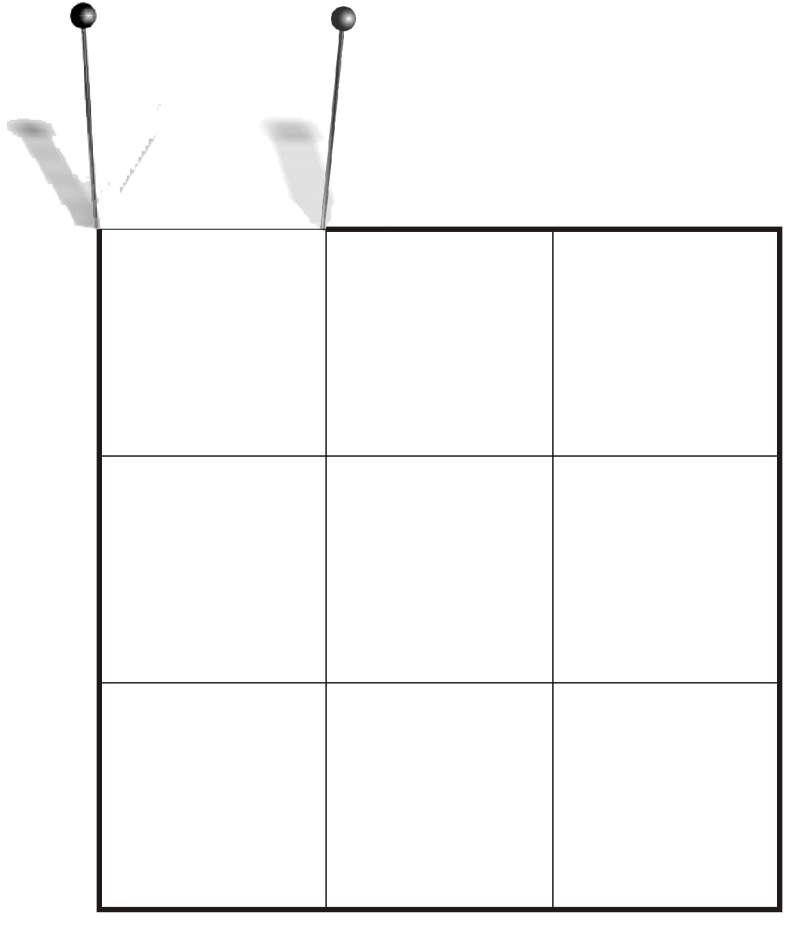
...because the volume of a cube cannot be doubled using compass and ruler alone, this problem (also known as the *Delian problem*) has in the past been solved in various other ways.
Here a very simple and direct geometrical method is presented, involving the tight fit of a rectangle in a square with a gap in one of ist sides...

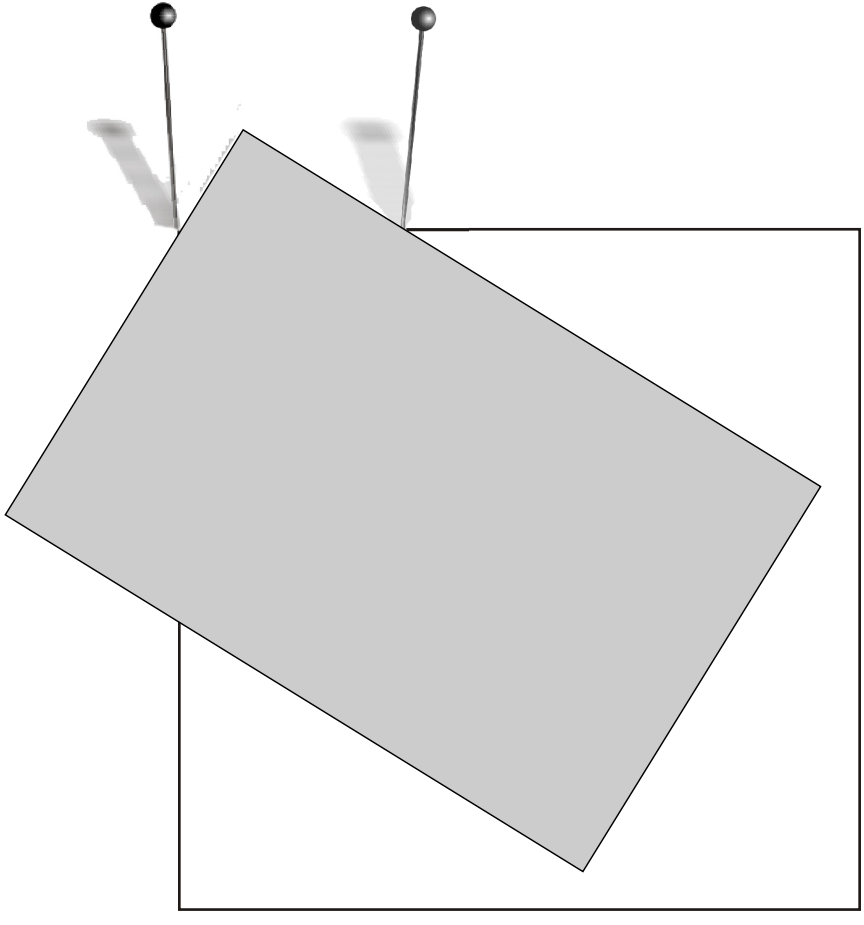


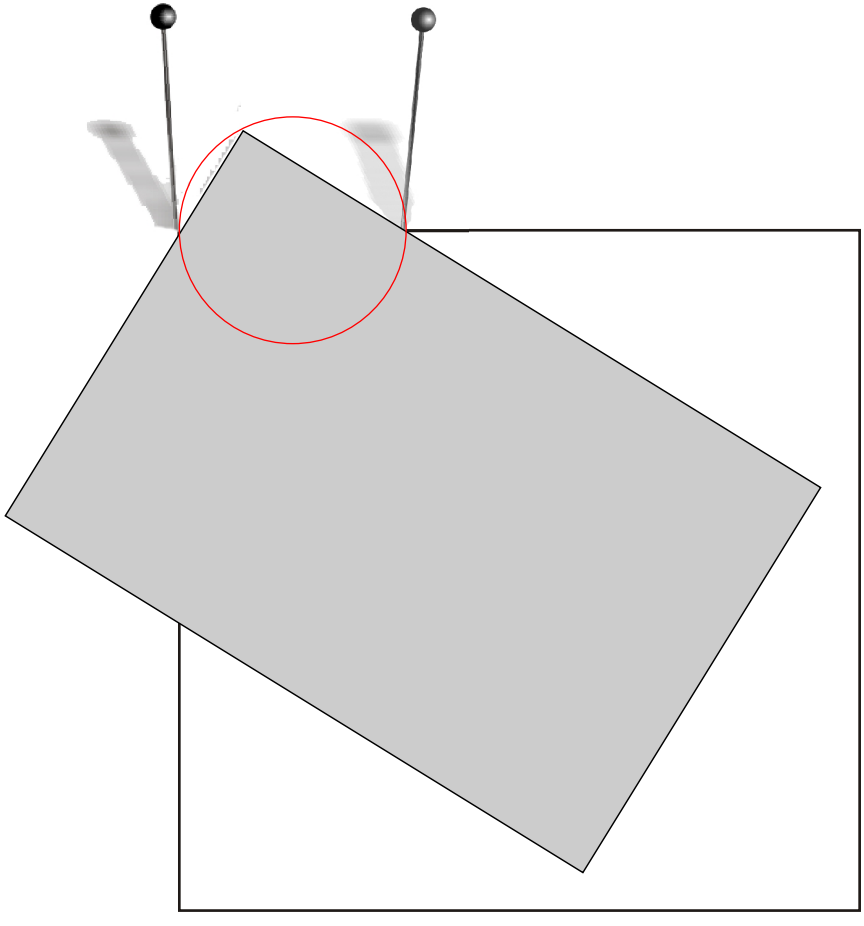


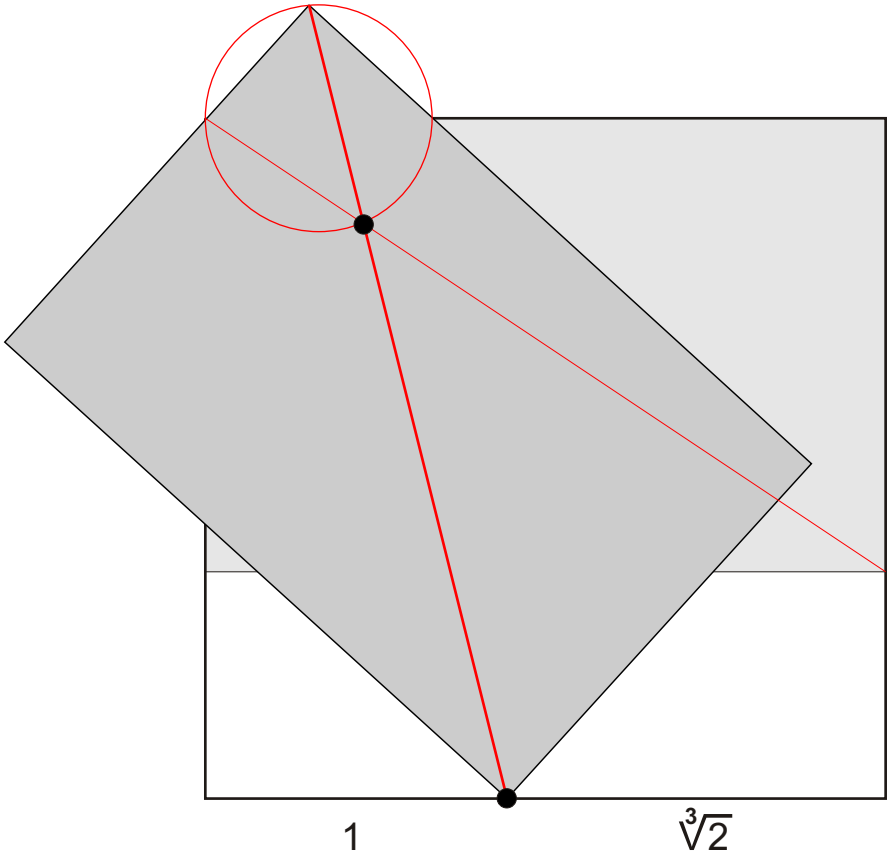








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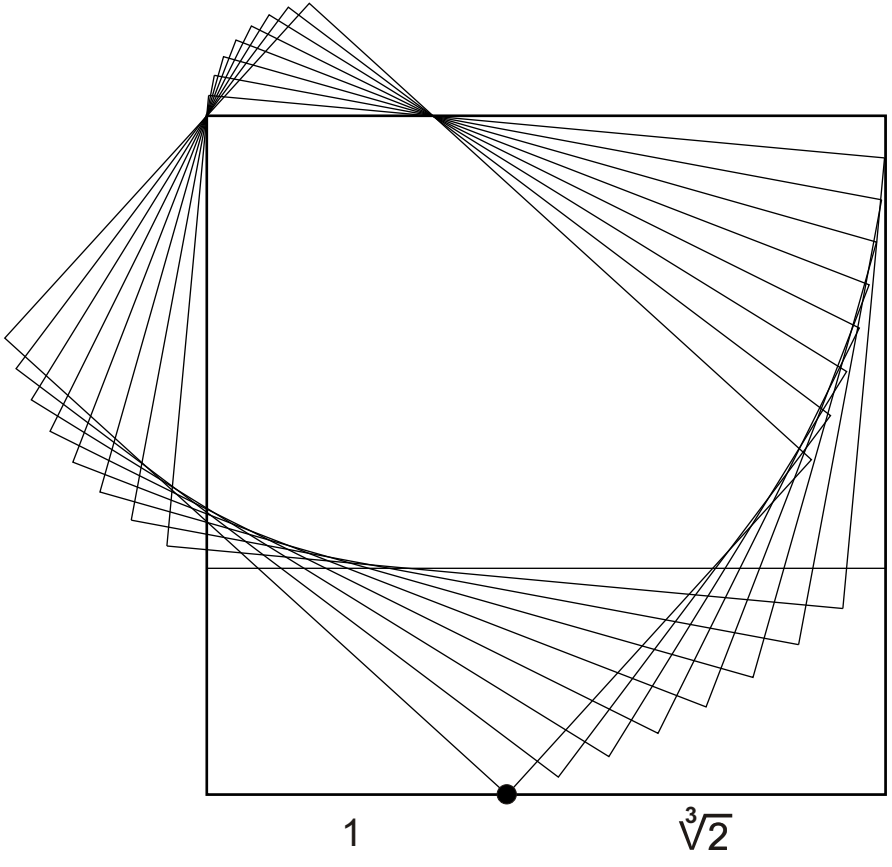
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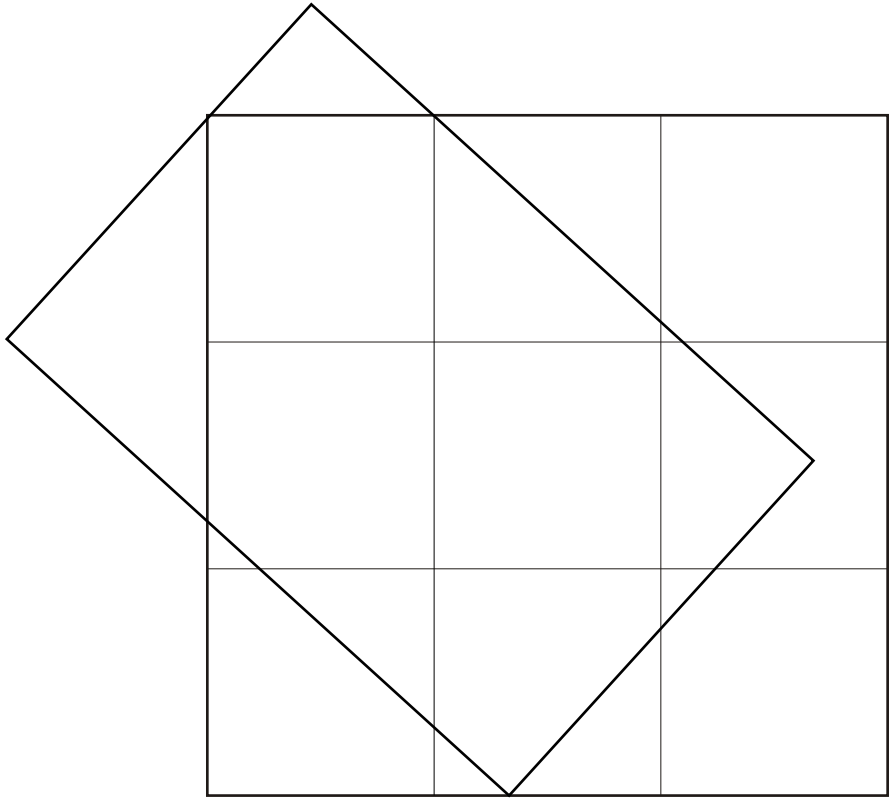
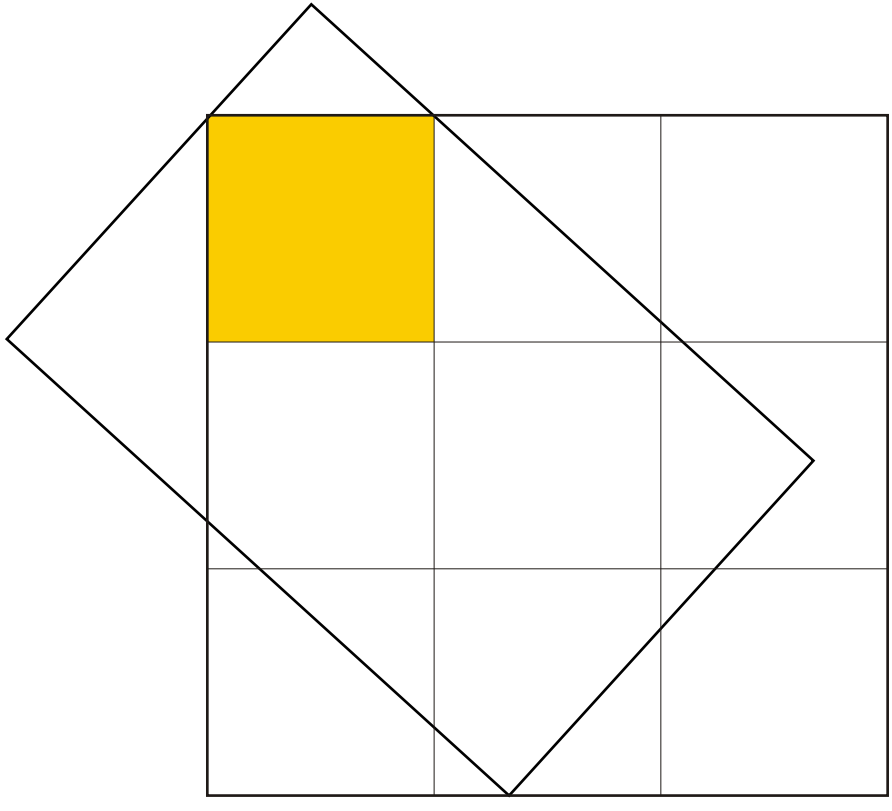
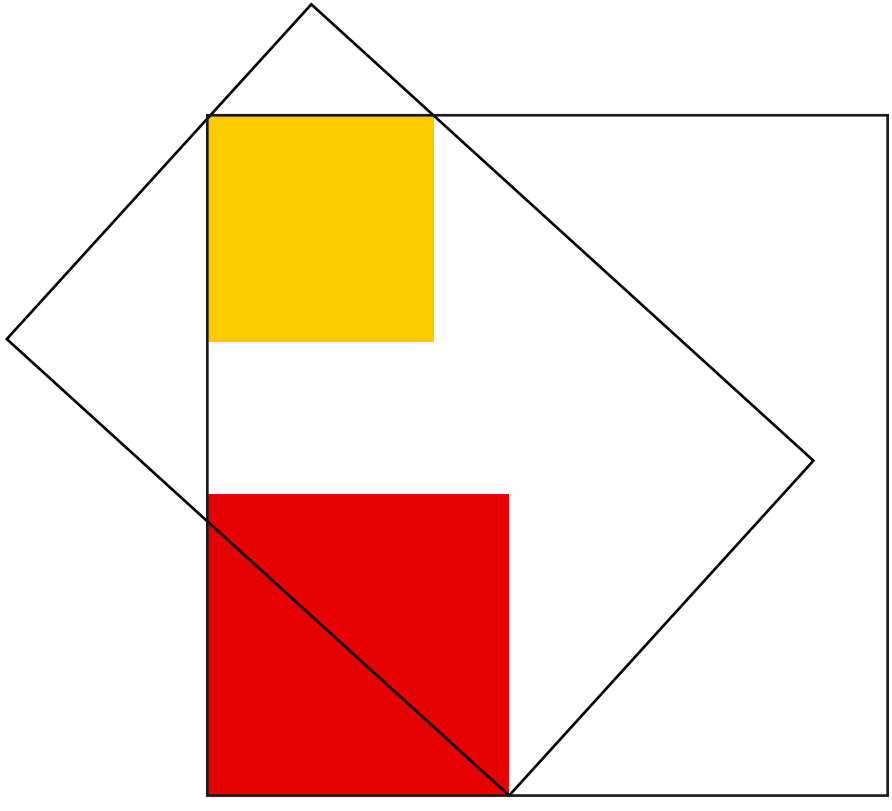
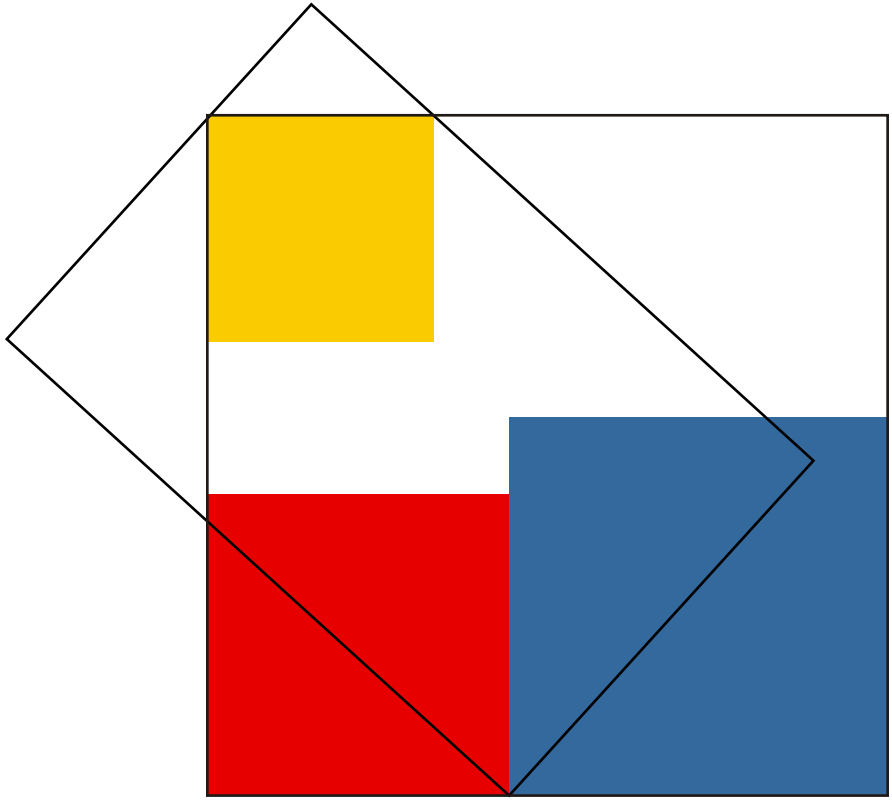
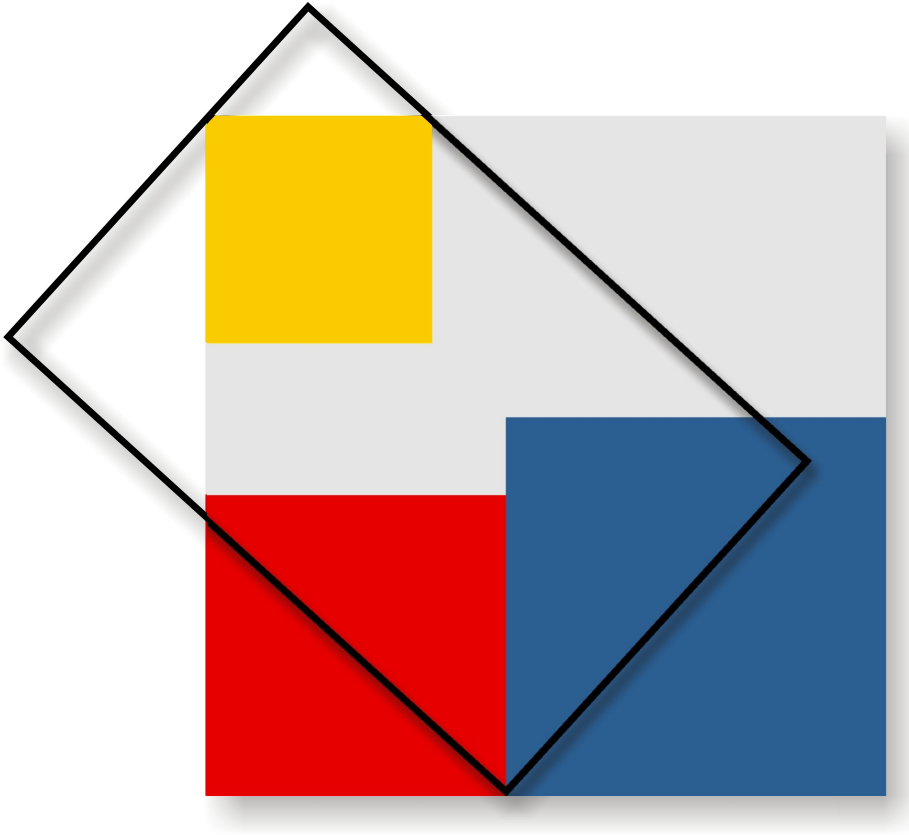


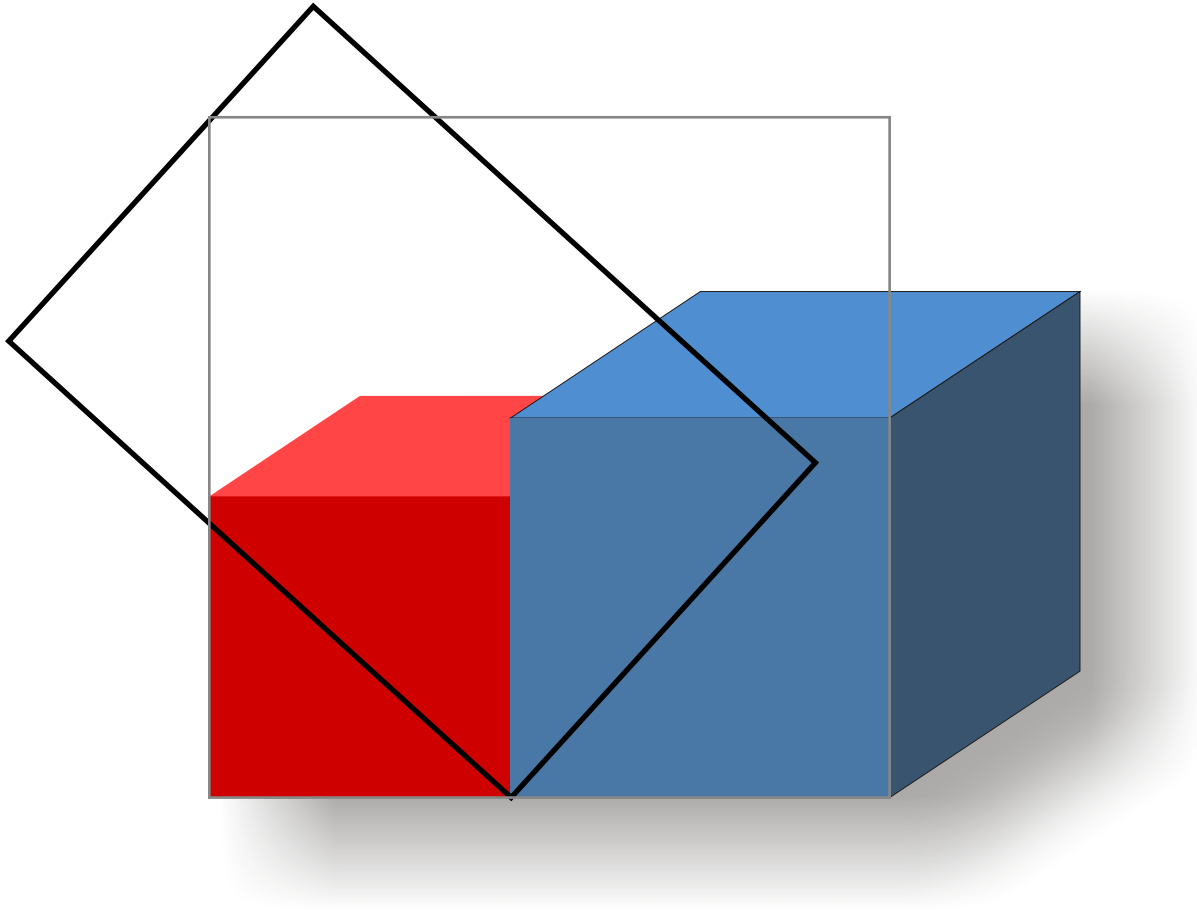


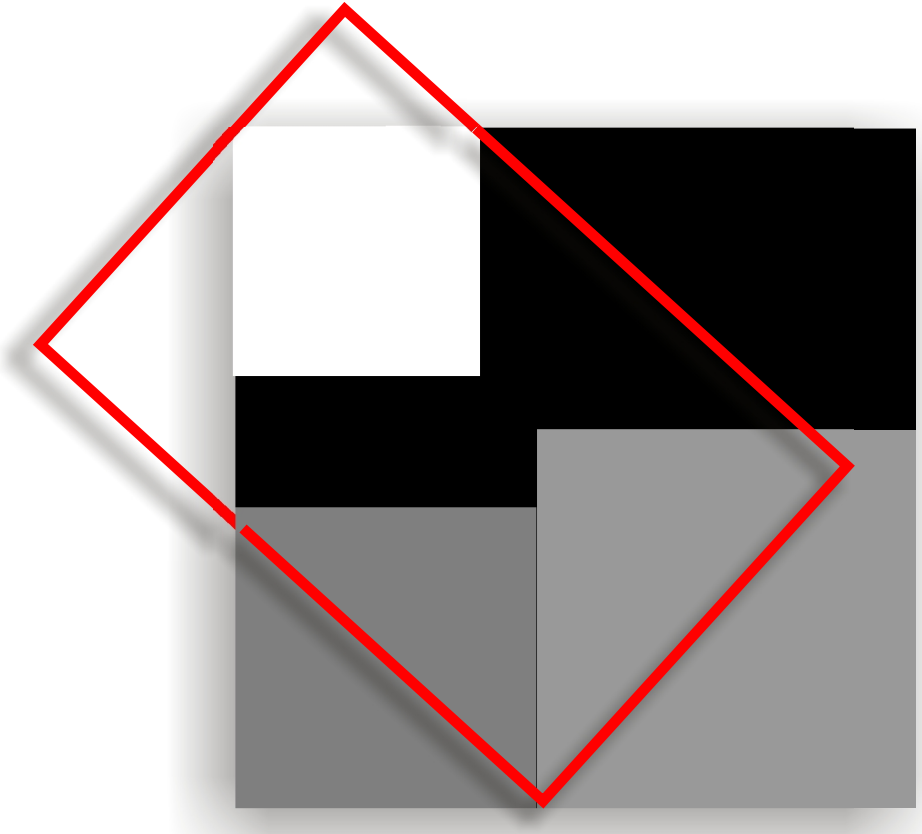






Jo Niemeyer 2020 Metal and Acrylic on canvas





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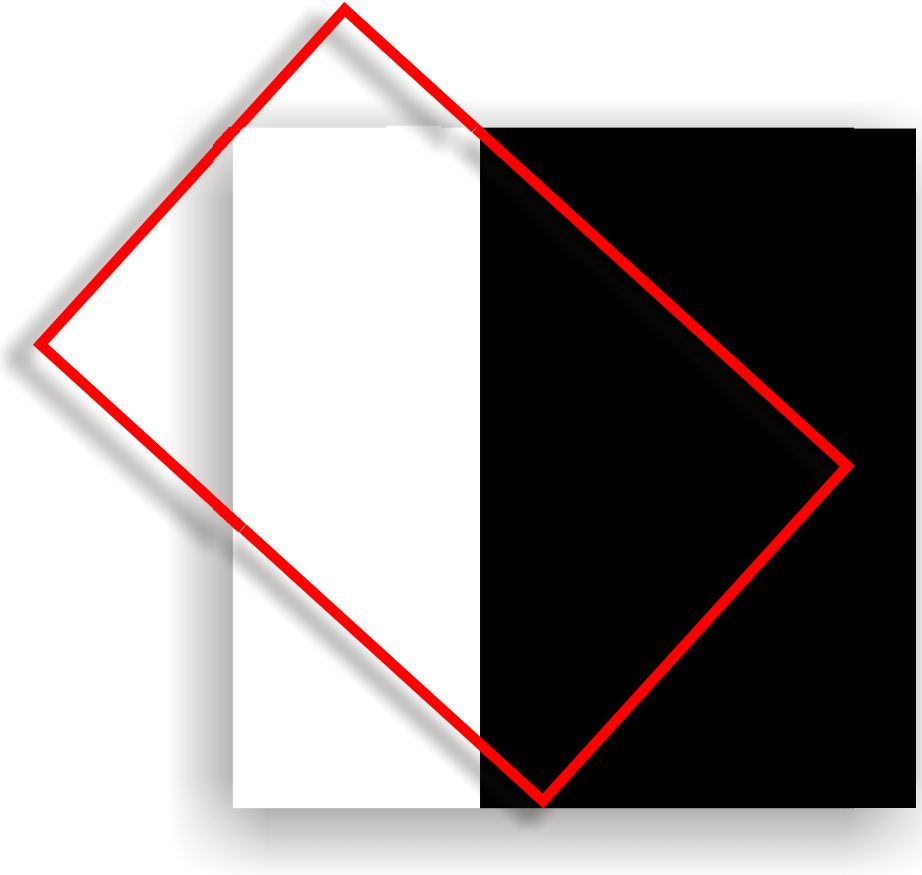

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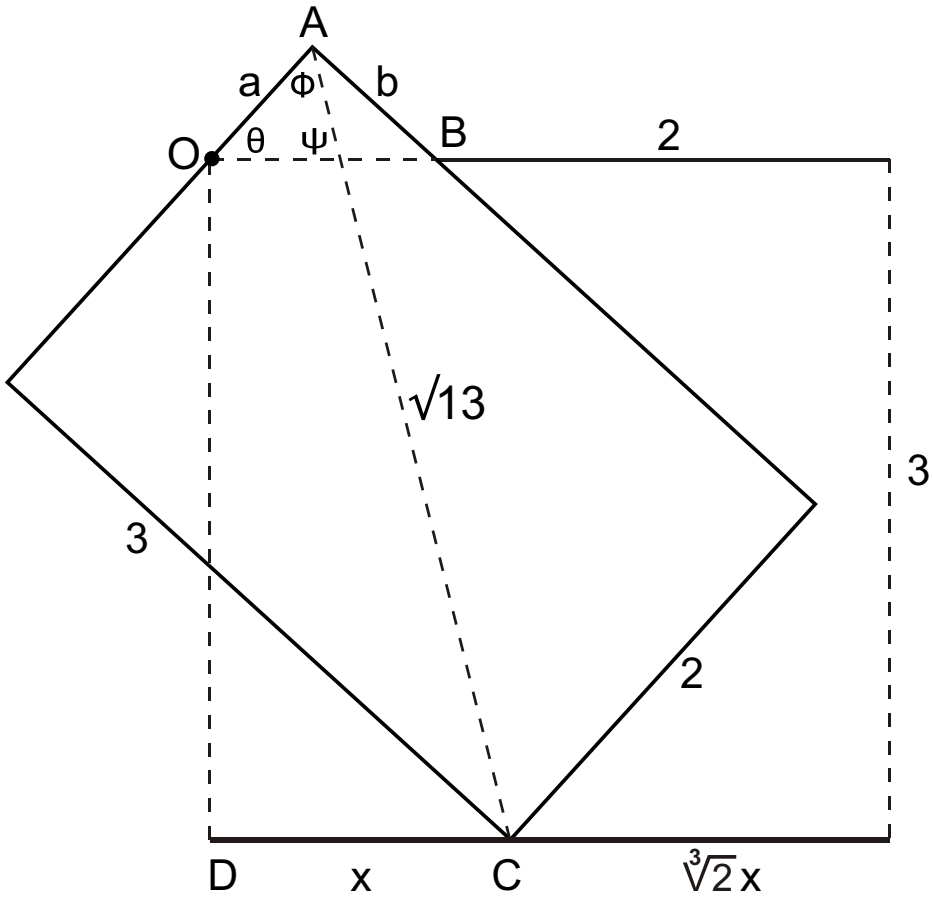
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# Doubling the Cube—Revisited

Jo Niemeyer[1] and Rabe R. von Randow[2]

[1]Freelance artist, Berlin; jo@niemeyer.info
[2]Retired, Formerly University of Bonn; rvrandow@t-online.de

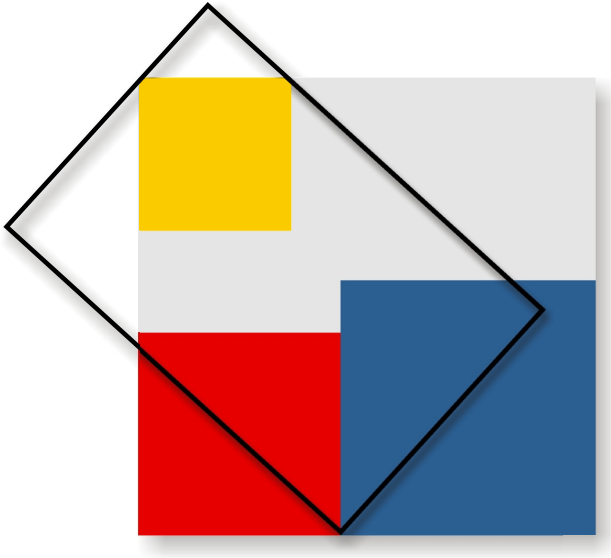
## Abstract

Constructing a cube with twice the volume of a given cube is one of the three famous problems of Greek antiquity which resists all attempts using ruler and compass only. The other two are trisecting an angle and squaring the circle. The impossibility of all three constructions with compass and straightedge has been proved, but many novel geometric methods have been devised to solve them by breaking the Greek rules. Here the first author presents a method which he has devised for doubling the cube, and the second author presents a proof for its correctness.

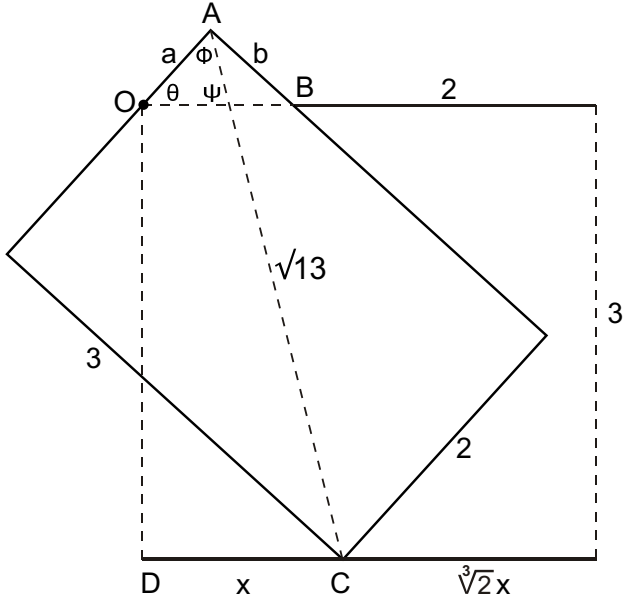
Whenever "doubling the cube" is mentioned, mathematicians usually beat a hasty retreat. Of course it is common knowledge that it cannot be done by ruler and compass alone, but there are many very subtle geometric methods to achieve it.

The first author is a German graphic artist and painter in the direction of concrete art [2, 3]. One day he showed me, the second author, a method he had devised for doubling the cube, shown in Figure 1a [1].

The construction involves sliding a $2 \times 3$ rectangle inside a $3 \times 3$ square where, at first, the only constraints are that the rectangle must touch the top left vertex as well as a point on the top edge that is 2 units from the top right vertex, as if there were a too-small "door" through which the rectangle were trying to pass. Sliding the rectangle until the vertex diagonal from the top one touches the bottom edge of the square, the point of contact divides the base into two pieces that have the proportion of the required cubes. He was convinced of



**(a)**




**(b)**

**Figure 1:** *Left: Niemeyer's painting, "Doubling the Cube." Right: A diagram to explain the proof that point C divides the base of a* $3 \times 3$ *rectangle into segments with the proportion* $1 : \sqrt[3]{2}$.

his construction but had no proof. I naturally had my doubts, but did not want to leave it at that and thought that it cannot be that difficult to prove or disprove. It turned out to be quite a lot of work, with features that, to my mind, merited making it known to others.

Let the vertices of the square be given in $\mathbb{R}^2$ by $(0,0)$, $(3,0)$, $(0,-3)$, and $(3,-3)$. Then the right-hand door post $B$ has coordinates $(1,0)$. The top right vertex $A$ of the rectangle is outside the door, the origin $O$ lies on one of its short edges, $B$ lies on one of its long edges, and the vertex $C$ diagonally opposite to $A$ lies on the bottom edge of the square, as shown in Figure 1b.

Let $C$ be given by $(x,-3)$ and denote $(0,-3)$ by $D$. Then $C$ divides the bottom edge of the square into two pieces of length $x$ and $3-x$. We claim that these pieces have the proportion required for cubes whose volumes are in the proportion 1:2. In other words, we need to show that $x\sqrt[3]{2} = 3 - x$.

As in the diagram, name legs of $\triangle OAB$ as $a$ and $b$, with angle $\angle AOB$ called $\theta$. Since the "door" has width 1, $a = \cos(\theta)$ and $b = \sin(\theta)$. The proof proceeds by analyzing vertical and horizontal components of the vector equation

$$\overline{OA} + \overline{AC} = \overline{OC}. \tag{1}$$

For the vertical components, name $\angle ACD$ as $\psi$ and $\angle OAC$ as $\phi$, noting that $\psi = \pi - \theta - \phi$, so that $\sin(\psi) = \sin(\theta + \phi)$ and $\cos(\psi) = -\cos(\theta + \phi)$. Equating the vertical components of (1) gives $a\sin(\theta) - \sqrt{13}\sin(\psi) = -3$. By the angle addition formula, together with $\sin(\phi) = 3/\sqrt{13}$ and $\cos(\phi) = 2/\sqrt{13}$, we get

$$ab - \sqrt{13}\left(\frac{2b}{\sqrt{13}} + \frac{3b}{\sqrt{13}}\right) = -3, \text{ or } ab - 2b - 3a = -3.$$

Replacing $b$ with $\sqrt{1-a^2}$, squaring to eliminate the radical, and dividing by the known factor $a - 1$ gives us

$$a^3 - 3a^2 + 9a - 5 = 0.$$

By Cardan's solution for the cubic, the unique real solution to this equation is $a = 1 + \omega - \omega^2$, where $\omega = \sqrt[3]{2}$. The appearance of $\omega$ suggests that we are on the right track. It follows that $b = \sqrt{1 - a^2} = 2 - \omega$.

Equating the horizontal components of (1) gives $a\cos(\theta) + \sqrt{13}\cos(\psi) = x$. Our formulas for $a$ and $b$ quickly give $x = a^2 - 2a + 3b = \omega^2 - \omega + 1$. To verify the desired equation, $x\sqrt[3]{2} = 3 - x$, we only need to verify that

$$(\omega^2 - \omega + 1)\omega = 3 - (\omega^2 - \omega + 1).$$

Therefore, the bottom edge of the square in Figure 1a is indeed divided into pieces with a ratio of $1 : \sqrt[2]{3}$, solving the problem of antiquity as claimed.

## References


[1] J. Niemeyer. "Doubling the Cube." Joint Mathematics Meetings Art Exhibition, 2021. http://gallery.bridgesmathart.org/exhibitions/2021-joint-mathematics-meetings/jo-niemeyer

[2] J. Niemeyer. Website. http://www.joniemeyer.com/jo_niemeyer/welcome.html

[3] I. Prinz, *Jo Niemeyer im Arithmeum.* Nicolaische Verlagsbuchhd, 2009.

[4] Bridges Conference 2021 https://www.bridgesmathart.org/b2021/

**Acknowledgement**

We are sorry to report that Rabe Rüdiger von Randow died between the time of submitting this article and its publication, so we dedicate this article to his memory.

He spent his youth in Shanghai and then in New Zealand. He studied mathematics, physics and chemistry at Auckland University and then went to Germany where he did a Ph.D. in algebraic topology with Professor F. Hirzebruch in Bonn. From 1972 until his retirement in 2001 he held a position in the Research Institute for Discrete Mathematics of the University of Bonn.

**Special Thanks**

Tuula
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Hans Walser